UNIT-II

Mathematical modeling of physical systems:

Mathematical modeling and transfer functions of electrical, mechanical and electro-mechanical elements - DC servo motors- two-phase AC servo motors - synchros.

MATHEMATICAL MODELS OF CONTROL SYSTEMS

A ***control system*** is a collection of physical objects (components) connected together to serve an objective. The input output relations of various physical components of a system are governed by ***differential equations.*** The mathematical model of a control system constitutes a set of differential equations. The response or output of the system can be studied by solving the differential equations for various input conditions.

The mathematical model of a system is linear if it obeys the principle of superposition a

Mathematical model of mechanical systems

Mechanical systems are classified as 1)mechanical translational systems

2) mechanical rotational systems

MECHANICAL TRANSLATIONAL SYSTEMS

The model of mechanical translational systems can be obtained by using three basic elements ***mass, spring and dash-pot.***

The weight of the mechanical system is represented by the element ***mass*** and it is assumed to be concentrated at the center of the body. The elastic deformation of the body can be represented by a ***spring.*** The friction existing in mechanical system can be represented by the ***dash-pot.***

When a force is applied to a translational mechanical system, it is opposed by opposing forces due to mass, friction and elasticity of the system. The force acting on a mechanical body are governed by ***Newton's second law of motion.*** For translational systems it states that the sum of forces acting on a body is zero. (or Newton's second law states that the sum of applied forces is equal to the sum of opposing forces on a body).

LIST OF SYMBOLS USED IN MECHANICAL TRANSLATIONAL SYSTEM

x = Displacement, m

$v = \frac{dx}{dt}$ = Velocity, m/sec

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ = Acceleration, m/sec^2

f = Applied force, N (Newtons)

f_m = Opposing force offered by mass of the body, N

f_k = Opposing force offered by the elasticity of the body (spring), N

f_b = Opposing force offered by the friction of the body (dash - pot), N

M = Mass, kg

K = Stiffness of spring, N/m

B = Viscous friction co-efficient, N-sec/m

FORCE BALANCE EQUATIONS OF IDEALIZED ELEMENTS

Consider an ideal mass element shown in fig 1.9 which has negligible friction and elasticity. Let a force be applied on it. The mass will offer an opposing force which is proportional to acceleration of the body.

Let, f = Applied force

f_m = Opposing force due to mass

Here, $f_m \propto \frac{d^2x}{dt^2}$ or $f_m = M\frac{d^2x}{dt^2}$

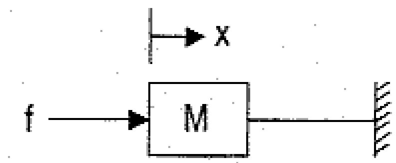


Fig 1.9 : Ideal mass element.

By Newton's second law, $\boxed{f = f_m = M\frac{d^2x}{dt^2}}$ .....(1.2)

Consider an ideal frictional element dashpot shown in fig 1.10 which has negligible mass and elasticity . Let a force be applied on it. The dash-pot will offer an opposing force which is proportional to velocity of the body.

Let, f = Applied force

f_b = Opposing force due to friction

Here, $f_b \propto \frac{dx}{dt}$ or $f_b = B\frac{dx}{dt}$

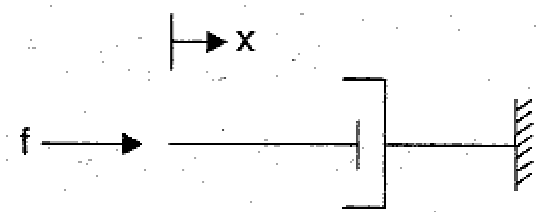


Fig 1.10 : Ideal dashpot with one end fixed to reference.

By Newton's second law, $\boxed{f = f_b = B\frac{dx}{dt}}$ .....(1.3)

When the dashpot has displacement at both ends as shown in fig 1.11, the opposing force is proportional to differential velocity.

$$f_b \propto \frac{d}{dt}(x_1 - x_2) \qquad \text{or} \qquad f_b = B\frac{d}{dt}(x_1 - x_2)$$

$$\therefore \boxed{f = f_b = B\frac{d}{dt}(x_1 - x_2)} \qquad .....(1.4)$$

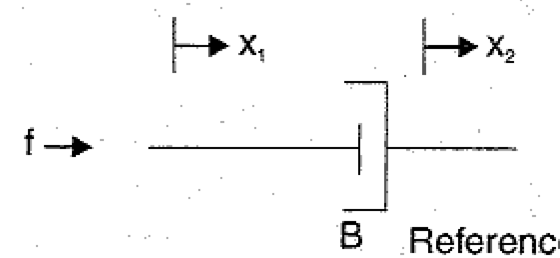


Fig 1.11 : Ideal dashpot with displacement at both ends.

Consider an ideal elastic element spring shown in fig 1.12, which has negligible mass and friction. Let a force be applied on it. The spring will offer an opposing force which is proportional to displacement of the body.

Let, f = Applied force

f_k = Opposing force due to elasticity

Here $f_k \propto x$ or $f_k = K\,x$

By Newton's second law, $\boxed{f = f_k = Kx}$ .....(1.5)

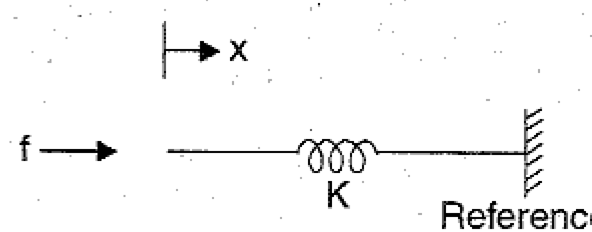


Fig 1.12 : Ideal spring with one end fixed to reference.

When the spring has displacement at both ends as shown in fig 1.13 the opposing force is proportional to differential displacement.

$$f_k \propto (x_1 - x_2) \qquad \text{or} \qquad f_k = K(x_1 - x_2)$$

$$\therefore \boxed{f = f_k = K(x_1 - x_2)} \qquad .....(1.6)$$

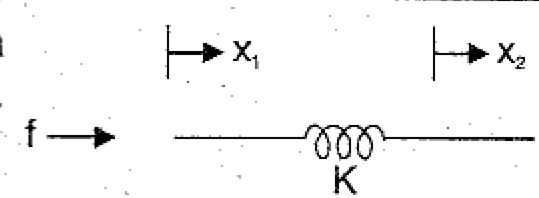
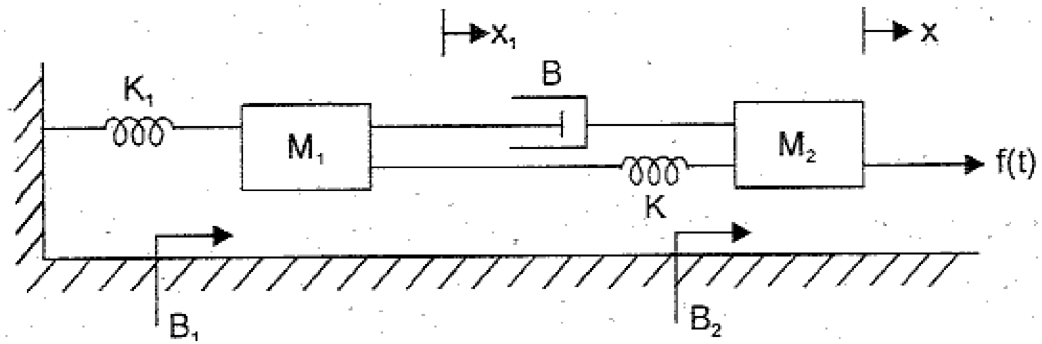


Fig 1.13 : Ideal spring with displacement at both ends.

Write the differential equations governing the mechanical system shown in fig 1. and determine the transfer function.



The free body diagram of mass M_1 is shown in fig

The opposing forces acting on mass M_1 are marked as f_{m1}, f_{b1}, f_b, f_{k1} and f_k.

$$f_{m1} = M_1 \frac{d^2x_1}{dt^2}; \qquad f_{b1} = B_1 \frac{dx_1}{dt}; \qquad f_{k1} = K_1 x_1;$$

$$f_b = B\frac{d}{dt}(x_1 - x); \quad f_k = K(x_1 - x)$$

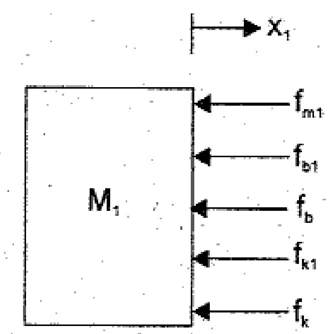


Fig 2 : Free body diagram of mass M_1 (node 1).

By Newton's second law,

$$f_{m1} + f_{b1} + f_b + f_{k1} + f_k = 0$$

$$\therefore M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B\frac{d}{dt}(x_1 - x) + K_1 x_1 + K(x_1 - x) = 0$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$M_1 s^2 X_1(s) + B_1 s X_1(s) + Bs\,[X_1(s) - X(s)] + K_1 X_1(s) + K\,[X_1(s) - X(s)] = 0$$

$$X_1(s)\,[M_1 s^2 + (B_1 + B)s + (K_1 + K)] - X(s)\,[Bs + K] = 0$$

$$X_1(s)\,[M_1 s^2 + (B_1 + B)s + (K_1 + K)] = X(s)\,[Bs + K]$$

$$\therefore X_1(s) = X(s)\,\frac{Bs + K}{M_1 s^2 + (B_1 + B)\,s + (K_1 + K)} \qquad \text{......(1}$$

The free body diagram of mass M_2 is shown in fig 3. The opposing forces acting on M_2 are marked as f_{m2}, f_{b2}, f_b and f_k.

$$f_{m2} = M_2 \frac{d^2x}{dt^2} \quad ; \quad f_{b2} = B_2 \frac{dx}{dt}$$

$$f_b = B\frac{d}{dt}(x - x_1) \quad ; \quad f_k = K(x - x_1)$$

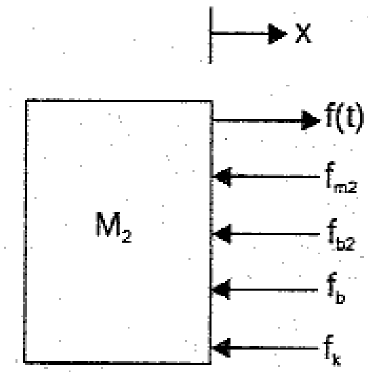


Fig 3 : Free body diagram of mass M_2 (node 2).

By Newton's second law,

$$f_{m2} + f_{b2} + f_b + f_k = f(t)$$

$$M_2 \frac{d^2x}{dt^2} + B_2 \frac{dx}{dt} + B\frac{d}{dt}(x - x_1) + K(x - x_1) = f(t)$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$M_2s^2X(s) + B_2sX(s) + Bs[X(s) - X_1(s)] + K[X(s) - X_1(s)] = F(s)$$

$$X(s)\,[M_2s^2 + (B_2 + B)s + K] - X_1(s)[Bs + K] = F(s) \quad .....(2)$$

Substituting for $X_1(s)$ from equation (1) in equation (2) we get,

$$X(s)\,[M_2s^2 + (B_2 + B)s + K] - X(s)\,\frac{(Bs + K)^2}{M_1s^2 + (B_1 + B)s + (K_1 + K)} = F(s)$$

$$X(s)\left[\frac{[M_2s^2+(B_2+B)s+K]\,[M_1s^2+(B_1+B)s+(K_1+K)]-(Bs+K)^2}{M_1s^2+(B_1+B)\,s+(K_1+K)}\right]=F(s)$$

$$\therefore \frac{X(s)}{F(s)}=\frac{M_1s^2+(B_1+B)s+(K_1+K)}{[M_1s^2+(B_1+B)s+(K_1+K)]\,[M_2s^2+(B_2+B)s+K]-(Bs+K)^2}$$

RESULT

The differential equations governing the system are,

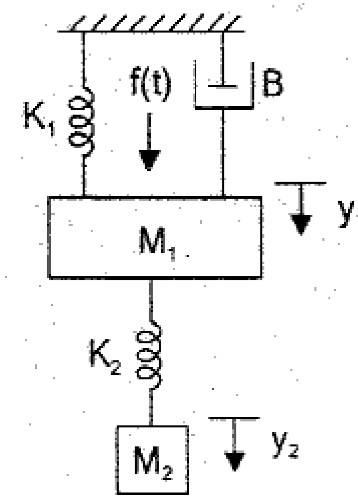
1. $M_1\frac{d^2x_1}{dt^2}+B_1\frac{dx_1}{dt}+B\frac{d}{dt}(x_1-x)+K_1x_1+K(x_1-x)=0$

2. $M_2\frac{d^2x}{dt^2}+B_2\frac{dx}{dt}+B\frac{d}{dt}(x-x_1)+K(x-x_1)=f(t)$

The transfer function of the system is,

$$\frac{X(s)}{F(s)}=\frac{M_1s^2+(B_1+B)\,s+(K_1+K)}{\left[M_1s^2+(B_1+B)\,s+(K_1+K)\right]\left[M_2s^2+(B_2+B)\,s+K\right]-(Bs+K)^2}$$

2) Determine the transfer function $\frac{Y_2(s)}{F(s)}$ of the system shown in fig



The free body diagram of mass M_1 is shown in fig 2.

The opposing forces are marked as f_{m1}, f_b, f_{k1} and f_{k2}

$$f_{m1} = M_1\frac{d^2y_1}{dt^2} \; ; \; f_b = B\frac{dy_1}{dt} \; ; \; f_{k1} = K_1y_1 \; ; \; f_{k2} = K_2(y_1 - y_2)$$

By Newton's second law, $f_{m1} + f_b + f_{k1} + f_{k2} = f(t)$

$$\therefore M_1\frac{d^2y_1}{dt^2} + B\frac{dy_1}{dt} + K_1y_1 + K_2(y_1 - y_2) = f(t) \qquad .....(1)$$

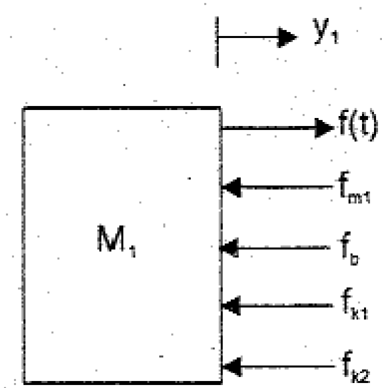


Fig 2.

On taking Laplace transform of equation (1) with zero initial condition we get,

$$M_1s^2Y_1(s) + BsY_1(s) + K_1Y_1(s) + K_2[Y_1(s) - Y_2(s)] = F(s)$$

$$Y_1(s)[M_1s^2 + Bs + (K_1 + K_2)] - Y_2(s)K_2 = F(s)$$

The free body diagram of mass M_2 is shown in fig 3. The opposing forces acting on M_2 are f_{m2} and f_{k2}.

$$f_{m2} = M_2 \frac{d^2y_2}{dt^2} \quad ; \qquad f_{k2} = K_2(y_2 - y_1)$$

By Newton's second law, $f_{m2} + f_{k2} = 0$

$$\therefore M_2 \frac{d^2y_2}{dt^2} + K_2(y_2 - y_1) = 0$$

On taking Laplace transform of above equation we get,

$$M_2s^2Y_2(s) + K_2[Y_2(s) - Y_1(s)] = 0$$

$$Y_2(s)\ [M_2s^2 + K_2] - Y_1(s)\ K_2 = 0$$

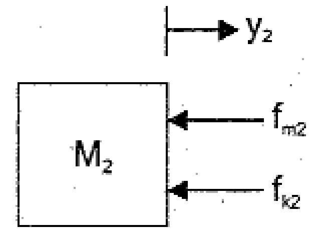


Fig 3.

$$\therefore Y_1(s) = Y_2(s)\ \frac{M_2s^2 + K_2}{K_2} \qquad \text{.....(3)}$$

Substituting for $Y_1(s)$ from equation (3) in equation (2) we get,

$$Y_2(s)\left[\frac{M_2s^2 + K_2}{K_2}\right]\left[M_1s^2 + Bs + (K_1 + K_2)\right] - Y_2(s)\,K_2 = F(s)$$

$$Y_2(s)\left[\frac{(M_2s^2 + K_2)\ [M_1s^2 + Bs + (K_1 + K_2)] - K_2^2}{K_2}\right] = F(s)$$

$$\therefore \frac{Y_2(s)}{F(s)} = \frac{K_2}{\left[M_1s^2 + Bs + (K_1 + K_2)\right]\left[M_2s^2 + K_2\right] - K_2^2}$$

RESULT

The differential equations governing the system are,

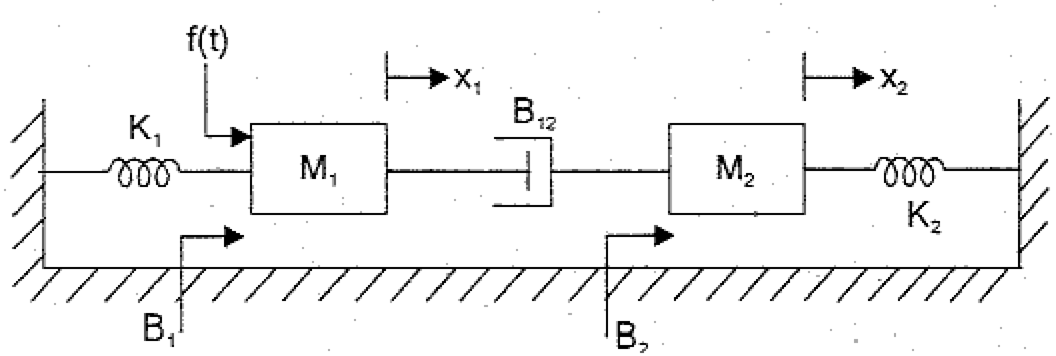
$$1.\quad M_1\frac{d^2y_1}{dt^2}+B\frac{dy_1}{dt}+K_1y_1+K_2(y_1-y_2)=f(t)$$

$$2.\quad M_2\frac{d^2y_2}{dt^2}+K_2(y_2-y_1)=0$$

The transfer function of the system is,

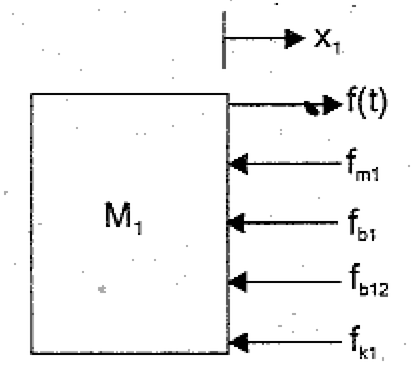
$$\frac{Y_2(s)}{F(s)}=\frac{K_2}{\left[M_1s^2+Bs+(K_1+K_2)\right]\left[M_2s^2+K_2\right]-K_2^2}$$

3) Determine the transfer function, $\frac{X_1(s)}{F(s)}$ and $\frac{X_2(s)}{F(s)}$ for the system shown in fig 1.



The free body diagram of mass M_1 is shown in fig 2.

The opposing forces are marked as f_{m1}, f_{b1}, f_{b12} and f_{k1}.



$$f_{m1} = M_1 \frac{d^2x_1}{dt^2} \quad ; \quad f_{b1} = B_1 \frac{dx_1}{dt} \quad ; \quad f_{b12} = B_{12} \frac{d}{dt}(x_1 - x_2) \quad ; \quad f_{k1} = K_1 x_1$$

By Newton's second law, $f_{m1} + f_{b1} + f_{b12} + f_{k1} = f(t)$

$$M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_{12} \frac{d(x_1 - x_2)}{dt} + K_1 x_1 = f(t)$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$M_1 s^2 X_1(s) + B_1 s\, X_1(s) + B_{12} s\, [X_1(s) - X_2(s)] + K_1 X_1(s) = F(s)$$

$$X_1(s)\, [M_1 s^2 + (B_1 + B_{12})\, s + K_1] - B_{12} s X_2(s) = F(s)$$

The free body diagram of mass M_2 is shown in fig 3. The opposing forces are marked as f_{m2}, f_{b2}, f_{b12} and f_{k2}.

$$f_{m2} = M_2 \frac{d^2x_2}{dt^2} \quad ; \quad f_{b2} = B_2 \frac{dx_2}{dt}$$

$$f_{b12} = B_{12} \frac{d}{dt}(x_2 - x_1) \; ; \quad f_{k2} = K_2 x_2$$

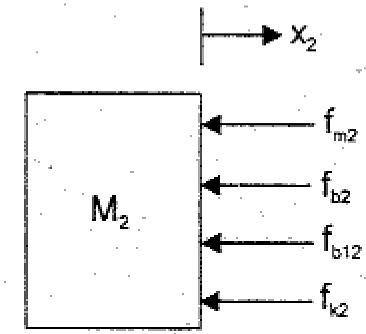


Fig 3.

By Newton's second law, $f_{m2} + f_{b2} + f_{b12} + f_{k2} = 0$

$$M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + B_{12} \frac{d(x_2 - x_1)}{dt} + K_2 x_2 = 0 \qquad .....(2)$$

On taking Laplace transform of equation (2) with zero initial conditions we get,

$$M_2 s^2 X_2(s) + B_2 s X_2(s) + B_{12} s\,[X_2(s) - X_1(s)] + K_2 X_2(s) = 0$$

$$X_2(s)\,[M_2 s^2 + (B_2 + B_{12})\,s + K_2] - B_{12}\,s\,X_1(s) = 0$$

$$X_2(s)\,[M_2 s^2 + (B_2 + B_{12})\,s + K_2] = B_{12}\,s\,X_1(s)$$

$$X_2(s) = \frac{B_{12} s\, X_1(s)}{[M_2 s^2 + (B_2 + B_{12})\,s + K_2]} \qquad .....(3)$$

Substituting for $X_2(s)$ from equation (3) in equation (1) we get,

$$X_1(s)\,[M_1 s^2 + (B_1 + B_{12})\,s + K_1] - \frac{(B_{12}s)^2\, X_1(s)}{M_2 s^2 + (B_2 + B_{12})\,s + K_2} = F(s)$$

$$\frac{X_1(s)\left[[M_1 s^2 + (B_1 + B_{12})\,s + K_1]\,[M_2 s^2 + (B_2 + B_{12})\,s + K_2] - (B_{12}s)^2\right]}{M_2 s^2 + (B_2 + B_{12})\,s + K_2} = F(s)$$

$$\therefore \frac{X_1(s)}{F(s)} = \frac{M_2 s^2 + (B_2 + B_{12})\,s + K_2}{[M_1 s^2 + (B_1 + B_{12})\,s + K_1]\,[M_2 s^2 + (B_2 + B_{12})\,s + K_2] - (B_{12}s)^2}$$

From equation (3) we get,

$$X_1(s) = \frac{[M_2s^2 + (B_2 + B_{12})\, s + K_2]\, X_2(s)}{B_{12}s} \qquad .....(4)$$

Substituting for $X_1(s)$ from equation (4) in equation (1) we get,

$$\frac{X_2(s)\, [M_2s^2 + (B_2 + B_{12})\, s + K_2]}{B_{12}s}\, [M_1s^2 + (B_1 + B_{12})\, s + K_1] - B_{12}s\, X_2(s) = F(s)$$

$$X_2(s) \left[\frac{[M_2s^2 + (B_2 + B_{12})\, s + K_2]\, [M_1s^2 + (B_1 + B_{12})\, s + K_1] - (B_{12}s)^2}{B_{12}s} \right] = F(s)$$

$$\therefore \frac{X_2(s)}{F(s)} = \frac{B_{12}s}{[M_2s^2 + (B_2 + B_{12})\, s + K_2]\, [M_1s^2 + (B_1 + B_{12})\, s + K_1] - (B_{12}s)^2}$$

RESULT

The differential equations governing the system are,

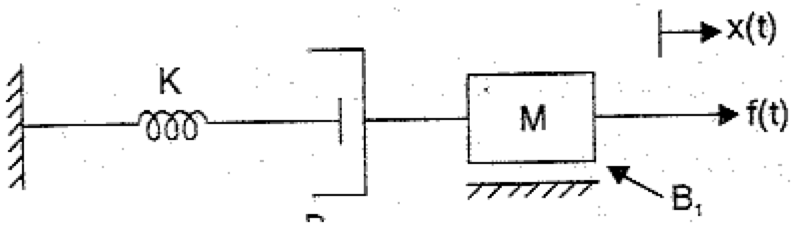
1. $M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_{12} \frac{d(x_1 - x_2)}{dt} + K_1x_1 = f(t)$
2. $M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + B_{12} \frac{d(x_2 - x_1)}{dt} + K_2x_2 = 0$

The transfer functions of the system are;

1. $\frac{X_1(s)}{F(s)} = \frac{M_2s^2 + (B_2 + B_{12})\, s + K_2}{[M_1s^2 + (B_1 + B_{12})\, s + K_1]\, [M_2s^2 + (B_2 + B_{12})\, s + K_2] - (B_{12}s)^2}$
2. $\frac{X_2(s)}{F(s)} = \frac{B_{12}s}{[M_2s^2 + (B_2 + B_{12})\, s + K_2]\, [M_1s^2 + (B_1 + B_{12})\, s + K_1] - (B_{12}s)^2}$

4)

Write the equations of motion in s-domain for the system shown in fig 1. Determine the transfer function of the system.



The free body diagram of mass M is shown in fig 2. The opposing forces are marked as f_m, f_{b1} and f_{b2}.

$$f_m = M\frac{d^2x}{dt^2} \quad ; \quad f_{b1} = B_1\frac{dx}{dt} \quad ; \quad f_{b2} = B_2\frac{d}{dt}(x - x_1)$$

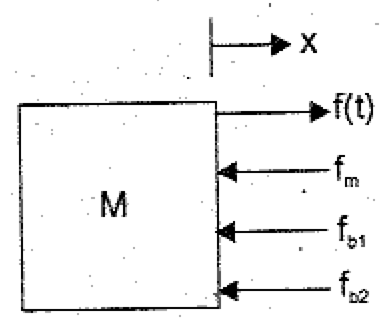


Fig 2.

By Newton's second law the force balance equation is,

$$f_m + f_{b1} + f_{b2} = f(t)$$

$$\therefore M\frac{d^2x}{dt^2} + B_1\frac{dx}{dt} + B_2\frac{d}{dt}(x - x_1) = f(t)$$

On taking Laplace transform of the above equation we get,

$$Ms^2 X(s) + B_1 s X(s) + B_2 s [X(s) - X_1(s)] = F(s)$$

$$\left[Ms^2 + (B_1 + B_2) s\right] X(s) - B_2 s X_1(s) = F(s) \qquad .....(1)$$

The free body diagram at the meeting point of spring and dashpot is shown in fig 3. The opposing forces are marked as f_k and f_{b2}.

$$f_{b2} = B_2 \frac{d}{dt}(x_1 - x) \; ; \quad f_k = K\, x_1$$

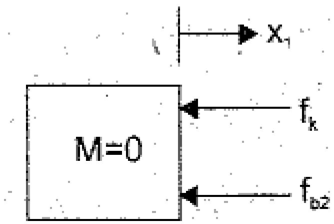


Fig 3.

By Newton's second law, $f_{b2} + f_k = 0$

$$\therefore B_2 \frac{d}{dt}(x_1 - x) + K\, x_1 = 0$$

On taking Laplace transform of the above equation we get,

$$B_2\, s\, [X_1(s) - X(s)] + K\, X_1(s) = 0$$

$$(B_2\, s + K)\, X_1(s) - B_2\, s\, X(s) = 0$$

$$\therefore X_1(s) = \frac{B_2\, s}{B_2\, s + K}\, X(s) \qquad .....(2)$$

Substituting for $X_1(s)$ from equation (2) in equation (1) we get,

$$\left[M\, s^2 + (B_1 + B_2)\, s\right] X(s) - B_2\, s \left[\frac{B_2\, s}{B_2\, s + K}\right] X(s) = F(s)$$

$$X(s)\; \frac{\left[[M\, s^2 + (B_1 + B_2)\, s]\,(B_2\, s + K) - (B_2\, s)^2\right]}{B_2\, s + K} = F(s)$$

$$\therefore \frac{X(s)}{F(s)} = \frac{B_2\, s + K}{[M\, s^2 + (B_1 + B_2)\, s]\,(B_2\, s + K) - (B_2\, s)^2}$$

RESULT

The differential equations governing the system are,

1. $M\frac{d^2x}{dt^2}+B_1\frac{dx}{dt}+B_2\frac{d}{dt}(x-x_1)=f(t)$

2. $B_2\frac{d}{dt}(x_1-x)+K\,x_1=0$

The equations of motion in s-domain are,

1. $[M\,s^2+(B_1+B_2)\,s]\,X(s)-B_2\,s\,X_1(s)=F(s)$

2. $(B_2\,s+K)\,X_1(s)-B_2\,s\,X(s)=0$

The transfer function of the system is,

$$\frac{X(s)}{F(s)}=\frac{B_2 s+K}{[M\,s^2+(B_1+B_2)\,s]\,(B_2\,s+K)-(B_2\,s)^2}$$

MECHANICAL ROTATIONAL SYSTEMS

The model of rotational mechanical systems can be obtained by using three elements, ***moment of inertia*** [J] of mass, ***dash-pot*** with rotational frictional coefficient [B] and ***torsional spring*** with stiffness [K].

The weight of the rotational mechanical system is represented by the moment of inertia of the mass.

The elastic deformation of the body can be represented by a spring (torsional spring).

The friction existing in rotational mechanical system can be represented by the dash-pot.

When a torque is applied to a rotational mechanical system, it is opposed by opposing torques due to moment of inertia, friction and elasticity of the system. The torques acting on a rotational mechanical body are governed by ***Newton's second law of motion*** for rotational systems. It states that the sum of torques acting on a body is zero (or Newton's law states that the sum of applied torques is equal to the sum of opposing torques on a body).

LIST OF SYMBOLS USED IN MECHANICAL ROTATIONAL SYSTEM

θ = Angular displacement, rad

$\frac{d\theta}{dt}$ = Angular velocity, rad/sec

$\frac{d^2\theta}{dt^2}$ = Angular acceleration, rad/sec^2

T = Applied torque, N-m

J = Moment of inertia, Kg-m^2/rad

B = Rotational frictional coefficient, N-m/(rad/sec)

K = Stiffness of the spring, N-m/rad

TORQUE BALANCE EQUATIONS OF IDEALISED ELEMENTS

Consider an ideal mass element shown in fig 1.14 which has negligible friction and elasticity. The opposing torque due to moment of inertia is proportional to the angular acceleration.

Let, T = Applied torque.

T_j = Opposing torque due to moment of inertia of the body.

Here $T_j \propto \frac{d^2\theta}{dt^2}$ or $T_j = J\frac{d^2\theta}{dt^2}$

By Newton's second law,

$$\boxed{T = T_j = J\frac{d^2\theta}{dt^2}} \qquad .....(1.7)$$

Fig 1.14 : Ideal rotational mass element

Consider an ideal frictional element dash pot shown in fig 1.15 which has negligible moment of inertia and elasticity. Let a torque be applied on it. The dash pot will offer an opposing torque which is proportional to the angular velocity of the body.

Let, T = Applied torque.

T_b = Opposing torque due to friction.

$T_b \propto \frac{d\theta}{dt}$ or $T_b = B\frac{d\theta}{dt}$

By Newton's second law, $$\boxed{T = T_b = B\frac{d\theta}{dt}} \qquad .....(1.8)$$

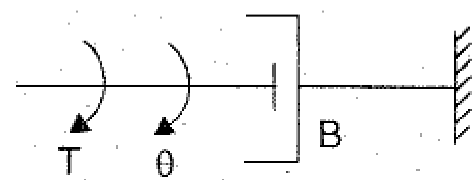


Fig 1.15 : Ideal rotational dash-pot with one end fixed to reference.

When the dash pot has angular displacement at both ends as shown in fig 1.16, the opposing torque is proportional to the differential angular velocity.

$$T_b \propto \frac{d}{dt}(\theta_1 - \theta_2) \quad \text{or} \quad T_b = B\frac{d}{dt}(\theta_1 - \theta_2)$$

$$\boxed{\therefore T = T_b = B\frac{d}{dt}(\theta_1 - \theta_2)} \qquad \text{.....(1.9)}$$

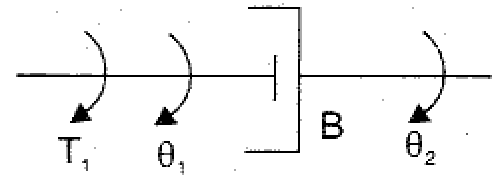


Fig 1.16 : Ideal dash-pot with angular displacement at both ends.

Consider an ideal elastic element, torsional spring as shown in fig 1.17, which has negligible moment of inertia and friction. Let a torque be applied on it. The torsional spring will offer an opposing torque which is proportional to angular displacement of the body.

Let, T = Applied torque.

T_k = Opposing torque due to elasticity.

$$T_k \propto \theta \quad \text{or} \quad T_k = K\theta$$

By Newton's second law, $\boxed{T = T_k = K\theta}$ .....(1.10)

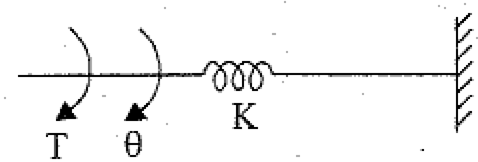


Fig 1.17 : Ideal spring with one end fixed to reference.

When the spring has angular displacement at both ends as shown in fig 1.18 the opposing torque is proportional to differential angular displacement.

$$T_k \propto (\theta_1 - \theta_2) \quad \text{or} \quad T_k = K(\theta_1 - \theta_2)$$

$$\therefore \boxed{T = T_k = K(\theta_1 - \theta_2)} \qquad \text{.....(1.11)}$$

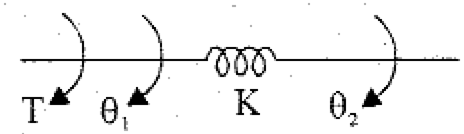
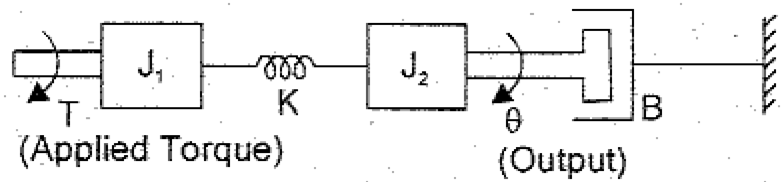


Fig 1.18 : Ideal spring with angular displacement at both ends.

1)

Write the differential equations governing the mechanical rotational system shown in fig 1. Obtain the transfer functio of the system.



The free body diagram of J_1 is shown in fig

opposing torques acting on J_1 are marked as T_{j1} and T_k.

$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2} \quad ; \quad T_k = K(\theta_1 - \theta)$$

By Newton's second law, $T_{j1} + T_k = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta) = T$$

$$J_1 \frac{d^2\theta_1}{dt^2} + K\theta_1 - K\theta = T \qquad .....(1)$$

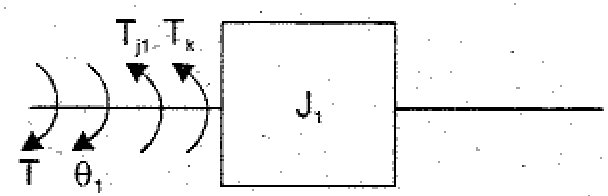


Fig 2 : *Free body diagram of mass with moment of inertia J_1.*

On taking Laplace transform of equation (1) with zero initial conditions we get,

$$J_1 s^2 \theta_1(s) + K\theta_1(s) - K\theta(s) = T(s)$$

$$(J_1 s^2 + K)\, \theta_1(s) - K\, \theta(s) = T(s) \qquad .....(2)$$

The free body diagram of mass with moment of inertia J_2 is shown in fig 3. The opposing torques acting on J_2 are marked as T_{j2}, T_b and T_k.

$$T_{j2} = J_2 \frac{d^2\theta}{dt^2} \quad ; \quad T_b = B\frac{d\theta}{dt} \quad ; \quad T_k = K(\theta - \theta_1)$$

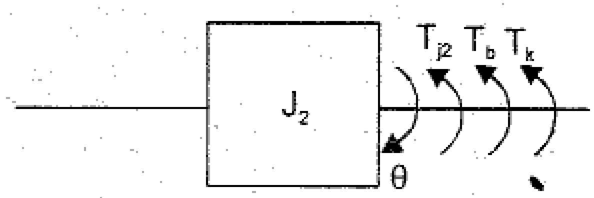


Fig 3 : Free body diagram of mass with moment of inertia J_2.

By Newton's second law, $T_{j2} + T_b + T_k = 0$

$$\therefore J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0$$

$$J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta - K\theta_1 = 0$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$J_2 s^2\theta(s) + B s\,\theta(s) + K\theta(s) - K\theta_1(s) = 0$$

$$(J_2 s^2 + Bs + K)\,\theta(s) - K\theta_1(s) = 0$$

$$\theta_1(s) = \frac{(J_2 s^2 + Bs + K)}{K}\,\theta(s) \qquad .....(3)$$

Substituting for $\theta_1(s)$ from equation (3) in equation (2) we get,

$$(J_1 s^2 + K)\,\frac{(J_2 s^2 + Bs + K)}{K}\,\theta(s) - K\theta(s) = T(s)$$

$$\left[\frac{(J_1 s^2 + K)\,(J_2 s^2 + Bs + K) - K^2}{K}\right]\theta(s) = T(s)$$

$$\therefore \frac{\theta(s)}{T(s)} = \frac{K}{(J_1 s^2 + K)\,(J_2 s^2 + Bs + K) - K^2}$$

RESULT

The differential equations governing the system are,

1. $J_1 \frac{d^2\theta_1}{dt^2} + K\theta_1 - K\theta = T$

2. $J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta - K\theta_1 = 0$

The transfer function of the system is,

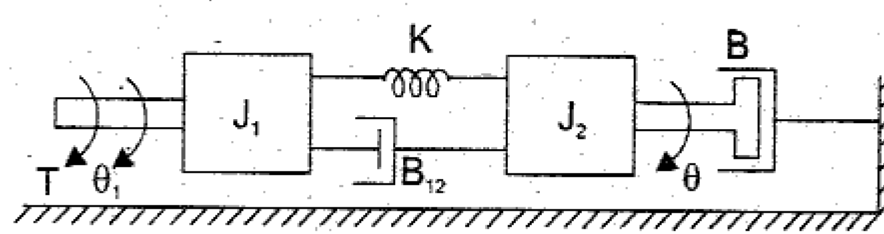
$$\frac{\theta(s)}{T(s)} = \frac{K}{(J_1s^2 + K)\,(J_2s^2 + Bs + K) - K^2}$$

2)

Write the differential equations governing the mechanical rotational system shown in fig 1. and determine the transfer function $\theta(s)/T(s)$.

SOLUTION

In the given system, the torque T is the input and the angular displacement θ is the output.



The free body diagram of J_1 is shown in fig

opposing torques acting on J_1 are marked as T_{j1}, T_{b12} and T_k.

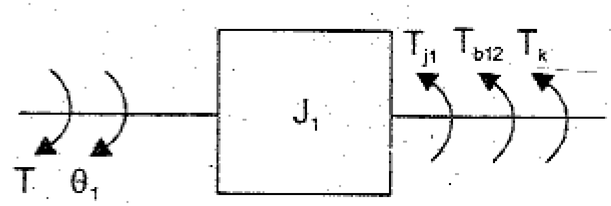


Fig 2 : Free body diagram of mass with moment of inertia J_1.

$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2} \quad ; \quad T_{b12} = B_{12}\frac{d}{dt}(\theta_1 - \theta) \quad ; \quad T_k = K(\theta_1 - \theta)$$

By Newton's second law, $T_{j1} + T_{b12} + T_k = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + B_{12}\frac{d}{dt}(\theta_1 - \theta) + K(\theta_1 - \theta) = T$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$J_1 s^2 \theta_1(s) + s\, B_{12}\,[\theta_1(s) - \theta(s)] + K\theta_1(s) - K\theta(s) = T(s)$$

$$\theta_1(s)\,[J_1 s^2 + sB_{12} + K] - \theta(s)\,[sB_{12} + K] = T(s) \qquad .....($$

The free body diagram of mass with moment of inertia J_2 is shown in fig 3. The opposing torques are marked as T_{j2}, T_t T_b and T_k.

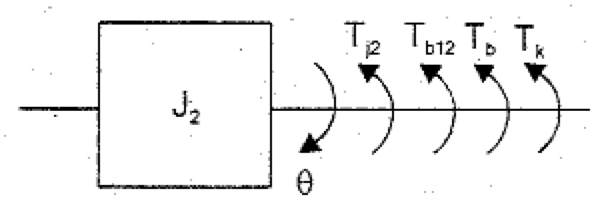


Fig 3 : Free body diagram of mass wi moment of inertia J_2.

$$T_{j2} = J_2 \frac{d^2\theta}{dt^2} \quad ; \quad T_{b12} = B_{12}\frac{d}{dt}(\theta - \theta_1)$$

$$T_b = B\frac{d\theta}{dt} \quad ; \quad T_k = K(\theta - \theta_1)$$

By Newton's second law, $T_{j2} + T_{b12} + T_b + T_k = 0$

$$J_2\frac{d^2\theta}{dt^2}+B_{12}\frac{d}{dt}(\theta-\theta_1)+B\frac{d\theta}{dt}+K(\theta-\theta_1)=0$$

$$J_2\frac{d^2\theta}{dt^2}-B_{12}\frac{d\theta_1}{dt}+\frac{d\theta}{dt}(B_{12}+B)+K\theta-K\theta_1=0$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$J_2s^2\theta(s)-B_{12}s\theta_1(s)+s\theta(s)\,[B_{12}+B]+K\theta(s)-K\theta_1(s)=0$$

$$\theta(s)\,[s^2J_2+s(B_{12}+B)+K]-\theta_1(s)\,[sB_{12}+K]=0$$

$$\theta_1(s)=\frac{[s^2J_2+s(B_{12}+B)+K]}{[sB_{12}+K]}\,\theta(s) \qquad \ldots$$

Substituting for $\theta_1(s)$ from equation (2) in equation (1) we get,

$$[J_1s^2+sB_{12}+K]\,\frac{[J_2s^2+s(B_{12}+B)+K]\,\theta(s)}{(sB_{12}+K)}-(sB_{12}+K)\,\theta(s)=T(s)$$

$$\left[\frac{(J_1s^2+sB_{12}+K)\,[J_2s^2+s(B_{12}+B)+K]-(sB_{12}+K)^2}{(sB_{12}+K)}\right]\theta(s)=T(s)$$

$$\therefore\frac{\theta(s)}{T(s)}=\frac{(sB_{12}+K)}{(J_1s^2+sB_{12}+K)\,[J_2s^2+s(B_{12}+B)+K]-(sB_{12}+K)^2}$$

RESULT

The differential equations governing the system are,

1. $$J_1\frac{d^2\theta_1}{dt^2}+B_{12}\frac{d}{dt}(\theta_1-\theta)+K(\theta_1-\theta)=T$$

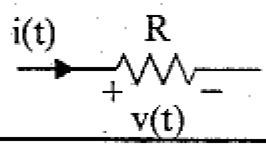
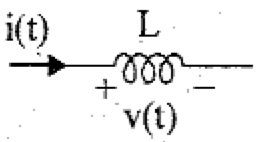
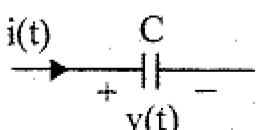
2. $$J_2\frac{d^2\theta}{dt^2}-B_{12}\frac{d\theta_1}{dt}+\frac{d\theta}{dt}(B_{12}+B)+K(\theta-\theta_1)=0$$

The transfer function of the system is,

$$\frac{\theta(s)}{T(s)}=\frac{(sB_{12}+K)}{(J_1s^2+sB_{12}+K)\,[J_2s^2+s(B_{12}+B)+K]-(sB_{12}+K)^2}$$

ELECTRICAL SYSTEMS

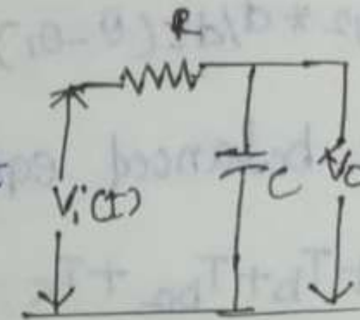
The models of electrical systems can be obtained by using resistor, capacitor and inductor. current-voltage relation of resistor, inductor and capacitor are given in table

Element	Voltage across the element	Current through the element
	$v(t) = Ri(t)$	$i(t) = \frac{v(t)}{R}$
	$v(t) = L\frac{d}{dt}i(t)$	$i(t) = \frac{1}{L}\int v(t)\,dt$
	$v(t) = \frac{1}{C}\int i(t)\,dt$	$i(t) = C\frac{dv(t)}{dt}$

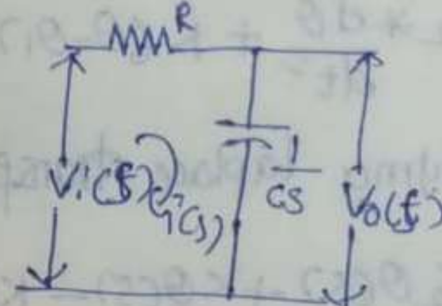
The differential equations governing the electrical systems can be formed by writing Kirchoff 's current law equations by choosing various nodes in the network or Kirchoff 's voltage law equations by choosing various closed paths in the network. The transfer function can be obtained by taking Laplace transform of the differential equations and rearranging them as a ratio of output to input.

The electrical circuits are genelally analysed by the application Kirchoff's voltage law and Kirchoff's cullent law.

1. Obtain the transfer function of the electrical network shown figure.



Sol. The laplace transform of above circuit is



$$V_i(s) = R * i(s) + \frac{1}{CS} * i(s)$$

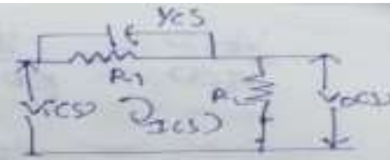
$$V_o(s) = \frac{1}{CS} * i(s)$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1/CS \; i(s)}{R\,i(s) + \frac{1}{CS}\,i(s)}$$

$$= \frac{\cancel{i(s)}}{(1+RCS)\cancel{i(s)}} = \frac{1}{1+RCS} \;//$$

$$V_i(s) = I(s) * \left(\frac{R_1 * 1/cs}{R_1 + 1/cs}\right) + R_2 I(s)$$

$$= I(s) * \frac{R_1}{1 + R_1 cs} + R_2 I(s)$$



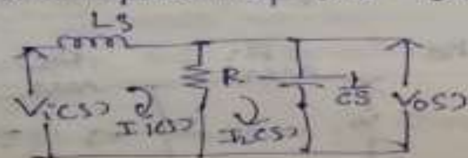
$$V_o(s) = R_2 \, I(s)$$

$$\frac{V_o(s)}{V_i(s)} = \frac{R_2 \, I(s)}{I(s)\left(\frac{R_1}{1+R_1cs} + R_2\right)} = \frac{R_2}{\frac{R_1}{1+R_1cs} + R_2} = \frac{R_2 + R_1R_2cs}{R_1 + R_2 + R_1R_2cs} \; //$$

$$= \frac{R_2(R_1cs + cs)}{R_1 + R_2 + R_1R_2cs} \; //$$

③ Obtain the transfer function of the given circuit :

Sol. The laplace transform of the above circuit is



$$V_o(s) = \frac{1}{cs} * I_2(s)$$

$$V_i(s) = Ls * I_1(s) + R(I_1(s) - I_2(s))$$

$$V_i(s) = I_1(s)(R + Ls) - R I_2(s) \rightarrow ①$$

$$0 = R * (I_2(s) - I_1(s)) + \frac{1}{cs}(I_2(s))$$

$$0 = -I_1(s)(R) + I_2(s)(R + 1/cs) \rightarrow ②$$

$$R(I_2(s) - I_1(s)) + I_2(s) \cdot 1/cs = 0$$

$$V_o(s) = 1/cs * I_2(s) \rightarrow ③$$

$$② \Rightarrow I_1(s) = \frac{I_2(s)}{R}(R + 1/cs)$$

$$① \Rightarrow \frac{I_2(s)}{R} * (R + Ls)\left(R + \frac{1}{cs}\right) - R\, I_2(s) = V_i(s)$$

$$\Rightarrow I_2(s)\left[\frac{(R+Ls)(R+1/cs)}{R} - R\right] = V_i(s)$$

$$\frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{Cs} * I_2(s)}{\left(\frac{(R+Ls)(R+\frac{1}{Cs})-R^2}{R}\right) I_2(s)}$$

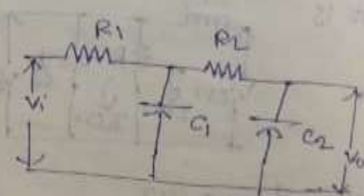
$$= \frac{\frac{R}{Cs}}{(R+Ls)(R+1/Cs)-R^2}$$

$$= \frac{R/Cs}{\frac{(R+Ls)(1+RCs)-R^2Cs}{Cs}}$$

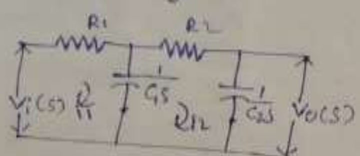
$$= \frac{R}{(R+Ls)(1+RCs)-R^2Cs} = \frac{R}{R+R^2Cs+Ls+RLCs^2-R^2Cs}$$

$$= \frac{R}{R+Ls+RLCs^2}$$

(3) Obtain the transfer function of the network shown below



The Laplace transform of the network is



$$V_i(s) = R_1 i_1(s) + \frac{1}{C_1 s}(i_1 - i_2)$$

$$V_i(s) = (R_1 + \frac{1}{C_1 s}) i_1(s) - \frac{1}{C_1 s} i_2(s) \rightarrow (1)$$

$$0 = R_2 i_2 + \frac{1}{C_2 s} i_2 + \frac{1}{C_1 s}(i_2(s) - i_1(s))$$

$$0 = -\frac{1}{C_1 s} i_1(s) + (R_2 + \frac{1}{C_2 s} + \frac{1}{C_1 s}) i_2(s) \rightarrow (2)$$

$$V_o(s) = \frac{1}{C_2 s} i_2(s)$$

$$(2) \Rightarrow i_1(s) = C_1(s)\, i_2(s) (R_2 + \frac{1}{C_2 s} + \frac{1}{C_1 s})$$

$$(1) \Rightarrow V_i(s) = (R_1 + \frac{1}{C_1 s}) i_2(s) (R_2 + \frac{1}{C_2 s} + \frac{1}{C_1 s}) - \frac{1}{C_1 s} i_2(s)$$

$$V_i(s) = i_2(s)\left[R_1 + \frac{1}{C_1 s} + R_2 + \frac{1}{C_2 s} + \cancel{\frac{1}{C_1 s}} - \cancel{\frac{1}{C_1 s}}\right]$$

$$V_i(s) = i_2(s)(R_1 + R_2 + \frac{1}{C_1 s} + \frac{1}{\ldots})$$

TRANSFER FUNCTION OF ARMATURE CONTROLLED DC MOTOR

The speed of DC motor is directly proportional to armature voltage and inversely proportional flux in field winding. In armature controlled DC motor the desired speed is obtained by varying t armature voltage. This speed control system is an electro-mechanical control system. The electric system consists of the armature and the field circuit but for analysis purpose, only the armature circuit considered because the field is excited by a constant voltage. The mechanical system consists of t rotating part of the motor and load connected to the shaft of the motor. The armature controlled D motor speed control system is shown in fig 1

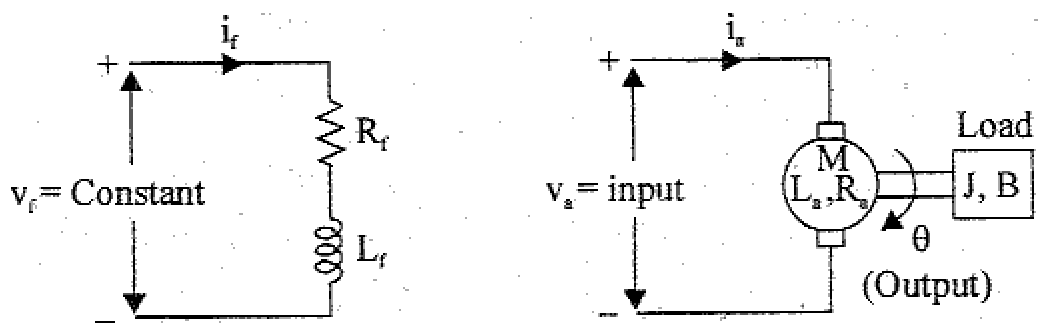


Fig 1 Armature controlled DC motor.

Let, R_a = Armature resistance, Ω

L_a = Armature inductance, H

i_a = Armature current, A

v_a = Armature voltage, V

e_b = Back emf, V

K_t = Torque constant, N-m/A

T = Torque developed by motor, N-m

θ = Angular displacement of shaft, rad

J = Moment of inertia of motor and load, Kg-m²/rad

B = Frictional coefficient of motor and load, N-m/(rad/sec)

K_b = Back emf constant, V/(rad/sec)

The equivalent circuit of armature is shown in fig 1.20.

By Kirchoff's voltage law, we can write,

$$i_a R_a + L_a \frac{di_a}{dt} + e_b = v_a \qquad .....(1.12)$$

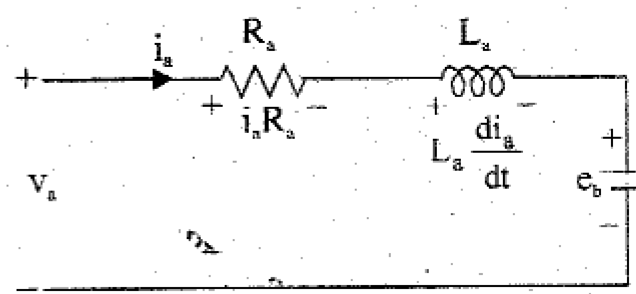


Fig . 2 : Equivalent circuit of armature.

The mechanical system of the motor is shown in fig 3

The differential equation governing the mechanical system of motor is given by

$$J\frac{d^2\theta}{dt^2}+B\frac{d\theta}{dt}=T \quad .....(1.14)$$

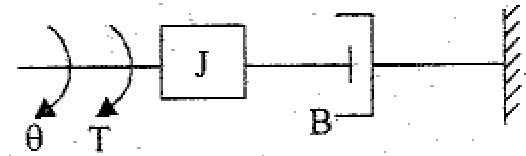


Fig 3

Torque of DC motor is proportional to the product of lux and current. Since flux is constant in this system, the orque is proportional to i_a alone.

$$T \propto i_a$$

$$\therefore \text{ Torque, } T = K_t\, i_a \quad .....(1.13)$$

The back emf of DC machine is proportional to speed (angular velocity) of shaft.

$$\therefore e_b \propto \frac{d\theta}{dt} \quad \text{or} \quad \text{Back emf, } e_b = K_b\frac{d\theta}{dt} \quad .....(1.15)$$

Taking Laplace transform of the above equations

$$I_a(s)\,R_a + L_a s I_a(s) + E_b(s) = V_a(s) \quad .....(1.16)$$

$$T(s) = K_t I_a(s) \quad .....(1.17)$$

$$Js^2\theta(s) + B\,s\,\theta(s) = T(s) \quad .....(1.18)$$

$$E_b(s) = K_b s\,\theta(s) \quad .....(1.19)$$

On equating equations (1.17) and (1.18) we get,

$$K_t I_a(s) = (Js^2 + Bs)\,\theta(s)$$

$$I_a(s) = \frac{(Js^2 + Bs)}{K_t}\,\theta(s) \qquad \text{.....(1.20)}$$

Equation (1.16) can be written as,

$$(R_a + sL_a)\,I_a(s) + E_b(s) = V_a(s) \qquad \text{.....(1.21)}$$

Substituting for $E_b(s)$ and $I_a(s)$ from equation (1.19) and (1.20) respectively in equation (1.21),

$$(R_a + sL_a)\,\frac{(Js^2 + Bs)}{K_t}\theta(s) + K_b s\,\theta(s) = V_a(s)$$

$$\left[\frac{(R_a + sL_a)\,(Js^2 + Bs) + K_b K_t s}{K_t}\right]\theta(s) = V_a(s)$$

The required transfer function is $\frac{\theta(s)}{V_a(s)}$

$$\therefore \frac{\theta(s)}{V_a(s)} = \frac{K_t}{(R_a + sL_a)(Js^2 + Bs) + K_b K_t s} \qquad .....(1.22$$

The transfer function of armature controlled dc motor can be expressed in another standard form

$$\frac{\theta(s)}{V_a(s)} = \frac{K_t}{(R_a + sL_a)(Js^2 + Bs) + K_b K_t s} = \frac{K_t}{R_a\left(\frac{sL_a}{R_a} + 1\right) Bs\left(1 + \frac{Js^2}{Bs}\right) + K_b K_t\, s}$$

$$= \frac{K_t / R_a B}{s\left[(1 + sT_a)(1 + sT_m) + \frac{K_b K_t}{R_a B}\right]}$$

where, $\frac{L_a}{R_a} = T_a$ = Electrical time constant

$\frac{J}{B} = T_m$ = Mechanical time constant

TRANSFER FUNCTION OF FIELD CONTROLLED DC MOTOR

The speed of a DC motor is directly proportional to armature voltage and inversely proportional to flux. In field controlled DC motor the armature voltage is kept constant and the speed is varied by varying the flux of the machine. Since flux is directly proportional to field current, the flux is varied by varying field current. The speed control system is an electromechanical control system. The electrical system consists of armature and field circuit but for analysis purpose, only field circuit is considered because the armature is excited by a constant voltage. The mechanical system consists of the rotating part of the motor and the load connected to the shaft of the motor. The field controlled DC motor speed control system is shown in fig 1

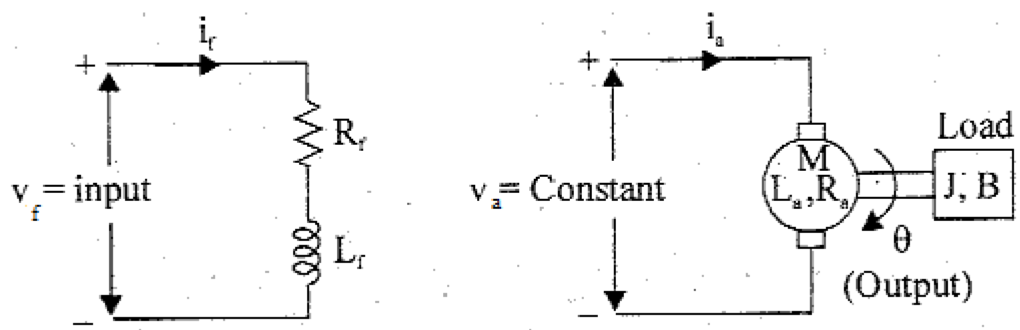


Fig 1 : Field controlled DC motor.

Let, R_f = Field resistance, Ω

L_f = Field inductance, H

i_f = Field current, A

v_f = Field voltage, V

T = Torque developed by motor, N-m

K_{tf} = Torque constant, N-m/A

J = Moment of inertia of rotor and load, Kg-m²/rad

B = Frictional coefficient of rotor and load, N-m/(rad/sec)

The equivalent circuit of field is shown in fig 2

By Kirchoff's voltage law, we can write

$$R_f i_f + L_f \frac{di_f}{dt} = v_f \qquad \text{..............1}$$

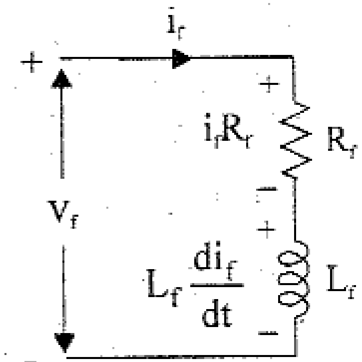


Fig 2 : Equivalent circuit of field.

The mechanical system of the motor is shown in fig 3 . The differential equation governing the mechanical system of the motor is given by,

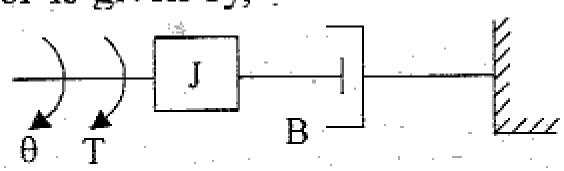


Fig.3

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \qquad \text{.................2}$$

The torque of DC motor is proportional to product of flux and armature current. Since armature current is constant in this system, the torque is proportional to flux alone, but flux is proportional to field current.

$$T \propto i_f, \quad \therefore \text{Torque, } T = K_{tf}\, i_f$$

On taking Laplace transform of the above equations with zero initial condition we get,

$$R_f I_f(s) + L_f s\, I_f(s) = V_f(s) \qquad \text{.................3}$$

$$K_{tf}\, I_f(s) = Js^2\theta(s) + Bs\theta(s) \qquad \text{.................4}$$

$$I_f(s) = s\,\frac{(Js+B)}{K_{tf}}\theta(s) \qquad \text{................5}$$

From equation 3

$$(R_f + sL_f)\, I_f(s) = V_f(s) \qquad \text{................6}$$

Substituting equ 5 in equ 6

$$(R_f + sL_f)s\,\frac{(Js+B)}{K_{tf}}\,\theta(s) = V_f(s)$$

$$\frac{\theta(s)}{V_f(s)} = \frac{K_{tf}}{s\,(R_f + sL_f)\,(B + sJ)}$$

$$= \frac{K_{tf}}{sR_f\left(1+\frac{sL_f}{R_f}\right)B\left(1+\frac{sJ}{B}\right)} = \frac{K_m}{s\,(1+sT_f)\,(1+sT_m)}$$

where, $K_m = \frac{K_{tf}}{R_f B}$ = Motor gain constant

$T_f = \frac{L_f}{R_f}$ = Field time constant

$T_m = \frac{J}{B}$ = Mechanical time constant

ELECTRICAL ANALOGOUS OF MECHANICAL TRANSLATIONAL SYSTEMS

Since the electrical systems has two types of inputs either voltage or current source, there are two types of analogies : ***force-voltage analogy*** and ***force-current analogy***.

FORCE-VOLTAGE ANALOGY

Item	Mechanical system	Electrical system (mesh basis system)
Independent variable (input)	Force, f	Voltage, e, v
Dependent variable (output)	Velocity, v	Current, i
	Displacement, x	Charge, q
Dissipative element	Frictional coefficient of dashpot, B	Resistance, R
Storage element	Mass, M	Inductance, L
	Stiffness of spring, K	Inverse of capacitance, 1/C
Physical law	Newton's second law $\sum f = 0$	Kirchoff's voltage law $\sum v = 0$

FORCE-CURRENT ANALOGY

Item	Mechanical system	Electrical system (node basis system)
Independent variable (input)	Force, f	Current, i
Dependent variable (output)	Velocity, v	Voltage, v
	Displacement, x	Flux, ϕ
Dissipative element	Frictional coefficient of dashpot, B	Conductance $G=1/R$
Storage element	Mass, M	Capacitance, C
	Stiffness of spring, K	Inverse of inductance, 1/L
Physical law	Newton's second law $\sum f = 0$	Kirchoff 's current law $\sum i = 0$

Write the differential equations governing the mechanical system shown in fig 1. Draw the force-voltage and force-current electrical analogous circuits and verify by writing mesh and node equations.

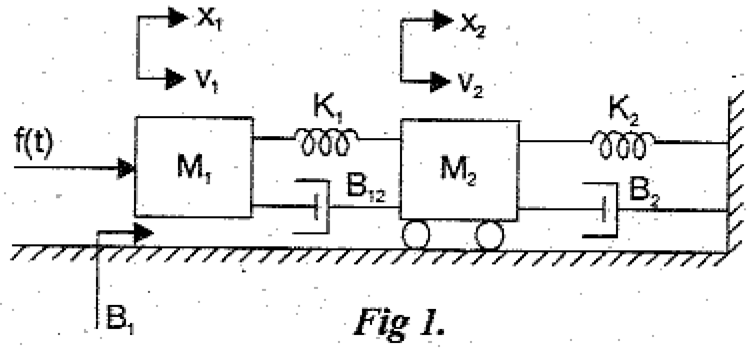


Fig 1.

The free body diagram of M_1 is shown in fig 2. The opposing forces are marked as f_{m1}, f_{b1}, f_{b12} and f_{k1}.

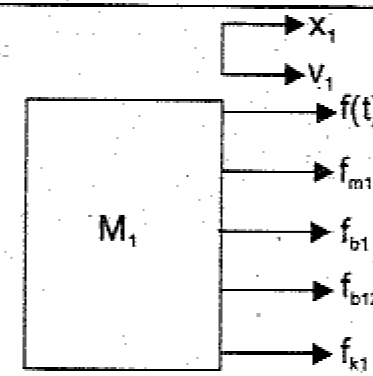


Fig 2.

$$f_{m1} = M_1 \frac{d^2x_1}{dt^2} \quad ; \quad f_{b1} = B_1 \frac{dx_1}{dt}$$

$$f_{b12} = B_{12} \frac{d}{dt}(x_1 - x_2) \quad ; \quad f_{k1} = K_1(x_1 - x_2)$$

By Newton's second law, $f_{m1} + f_{b1} + f_{b12} + f_{k1} = f(t)$

$$\therefore M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_{12} \frac{d}{dt}(x_1 - x_2) + K_1(x_1 - x_2) = f(t) \qquad ......(1)$$

The free body diagram of M_2 is shown in fig 3.The opposing forces are marked as f_{m2}, f_{b2}, f_{b12}, f_{k1} and f_{k2} .

$$f_{m2} = M_2 \frac{d^2x_2}{dt^2} \quad ; \quad f_{b2} = B_2 \frac{dx_2}{dt} \quad ; \quad f_{b12} = B_{12} \frac{d}{dt}(x_2 - x_1)$$

$$f_{k1} = K_1(x_2 - x_1) \quad ; \quad f_{k2} = K_2 x_2$$

By Newton's second law, $f_{m2} + f_{b2} + f_{k2} + f_{b12} + f_{k1} = 0$

$$M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + K_2 x_2 + B_{12} \frac{d}{dt}(x_2 - x_1) + K_1(x_2 - x_1) = 0 \qquad .....(2)$$

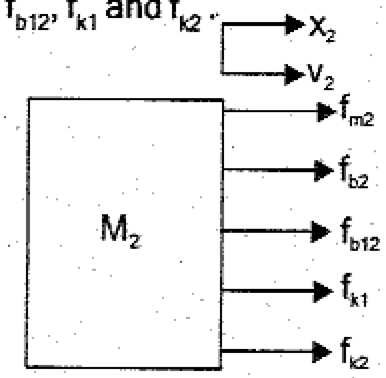


Fig 3.

On replacing the displacements by velocity in the differential equations (1) and (2) of the mechanical system we get,

$$\left(\text{i.e, } \frac{d^2x}{dt^2} = \frac{dv}{dt} \quad ; \quad \frac{dx}{dt} = v \text{ and } x = \int v\,dt \right)$$

$$M_1 \frac{dv_1}{dt} + B_1 v_1 + B_{12}(v_1 - v_2) + K_1 \int (v_1 - v_2)\, dt = f(t) \qquad .....(3)$$

$$M_2 \frac{dv_2}{dt} + B_2 v_2 + K_2 \int v_2\, dt + B_{12}(v_2 - v_1) + K_1 \int (v_2 - v_1)\, dt = 0 \qquad .....(4)$$

The electrical analogous elements for the elements of mechanical system are given below.

$f(t) \rightarrow e(t)$	$M_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
$v_1 \rightarrow i_1$	$M_2 \rightarrow L_2$	$B_2 \rightarrow R_2$	$K_2 \rightarrow 1/C_2$
$v_2 \rightarrow i_2$		$B_{12} \rightarrow R_{12}$	

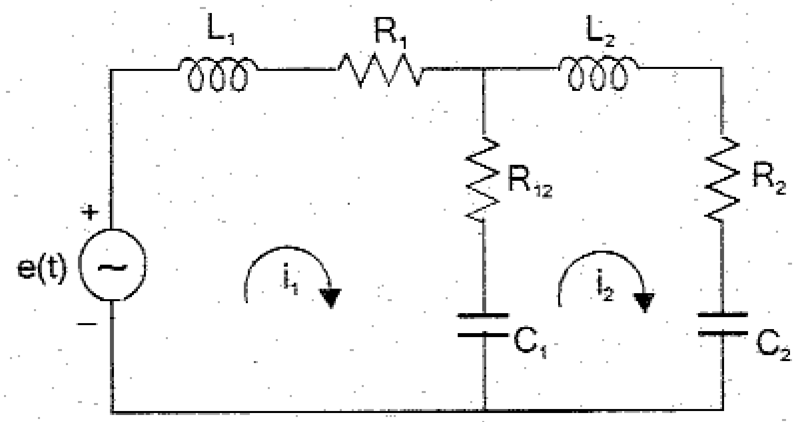


Fig 4 : Force-voltage electrical analogous circuit.

Fig 5.

Fig 6.

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 4 are given below (Refer fig 5 and 6).

$$L_1\frac{di_1}{dt} + R_1 i_1 + R_{12}(i_1 - i_2) + \frac{1}{C_1}\int (i_1 - i_2)dt = e(t) \qquad .....(5)$$

$$L_2\frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt + R_{12}(i_2 - i_1) + \frac{1}{C_1}\int (i_2 - i_1)dt = 0 \qquad .....(6)$$

It is observed that the mesh basis equations (5) and (6) are similar to the differential equations (3) and (4) governing the mechanical system.

FORCE-CURRENT ANALOGOUS CIRCUIT

The electrical analogous elements for the elements of mechanical system are given below.

$f(t) \rightarrow i(t)$	$M_1 \rightarrow C_1$	$B_1 \rightarrow 1/R_1$	$K_1 \rightarrow 1/L_1$
$v_1 \rightarrow v_1$	$M_2 \rightarrow C_2$	$B_2 \rightarrow 1/R_2$	$K_2 \rightarrow 1/L_2$
$v_2 \rightarrow v_2$	$B_{12} \rightarrow 1/R_{12}$		

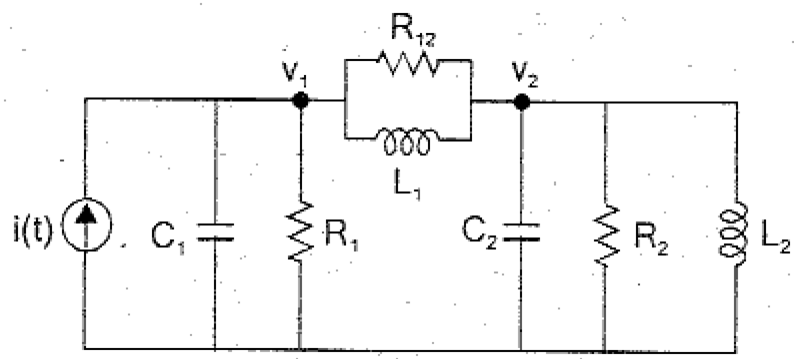


***Fig 7 :** Force-voltage electrical analogous circuit.*

The node basis equations using Kirchoff's current law for the circuit shown in fig 7 are given below (Refer fig 8 and 9).

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}v_1+\frac{1}{R_{12}}(v_1-v_2)+\frac{1}{L_1}\int(v_1-v_2)dt=i(t) \qquad .....(7)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_2}v_2+\frac{1}{L_2}\int v_2dt+\frac{1}{R_{12}}(v_2-v_1)+\frac{1}{L_1}\int(v_2-v_1)dt=0 \qquad .....(8)$$

It is observed that the node basis equations (7) and (8) are similar to the differential equations (3) and (4) governing the mechanical system.

ELECTRICAL ANALOGOUS OF MECHANICAL ROTATIONAL SYSTEMS

Since the electrical systems has two types of inputs either voltage source or current source, there are two types of analogies: ***torque-voltage analogy and torque-current analogy***.

TORQUE-VOLTAGE ANALOGY

Item	Mechanical rotational system	Electrical system (mesh basis system)
Independent variable (input)	Torque, T	Voltage, e, v
Dependent variable (output)	Angular Velocity, ω	Current, i
	Angular displacement, θ	Charge, q
Dissipative element	Rotational coefficient of dashpot, B	Resistance, R
Storage element	Moment of inertia, J	Inductance, L
	Stiffness of spring, K	Inverse of capacitance, 1/C
Physical law	Newton's second law $\Sigma T = 0$	Kirchoff's voltage law $\Sigma v = 0$

TORQUE-CURRENT ANALOGY

Item	Mechanical rotational system	Electrical system (node basis system)
Independent variable (input)	Torque, T	Current, i
Dependent variable (output)	Angular Velocity, ω	Voltage, v
	Angular displacement, θ	Flux, ϕ
Dissipative element	Rotational frictional coefficient of dashpot, B	Conductance, G = 1/R
Storage element	Moment of inertia, J	Capacitance, C
	Stiffness of spring, K	Inverse of inductance, 1/L
Physical law	Newton's second law $\sum T = 0$	Kirchoff 's current law $\sum i = 0$

Write the differential equations governing the mechanical rotational system shown in fig 1. Draw the torque-voltage and torque-current electrical analogous circuits and verify by writing mesh and node equations.

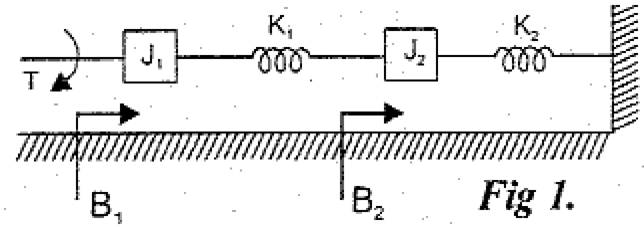


Fig 1.

SOLUTION

The free body diagram of J_1 is shown in fig 2. The opposing torques are marked as T_{j1}, T_{b1} and T_{k1}.

$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2} \quad ; \quad T_{b1} = B_1 \frac{d\theta_1}{dt} \quad ; \quad T_{k1} = K_1(\theta_1 - \theta_2)$$

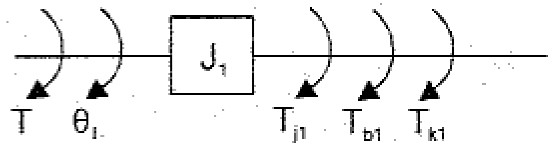


Fig 2.

By Newton's second law, $T_{j1} + T_{b1} + T_{k1} = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + K_1(\theta_1 - \theta_2) = T \qquad .....(1)$$

The free body diagram of J_2 is shown in fig 3. The opposing torques are marked as T_{j2}, T_{b2}, T_{k2} and T_{k1}.

$$T_{j2} = J_2 \frac{d^2\theta_2}{dt^2} \quad ; \quad T_{b2} = B_2 \frac{d\theta_2}{dt}$$

$$T_{k2} = K_2\theta_2 \quad ; \quad T_{k1} = K_1(\theta_2 - \theta_1)$$

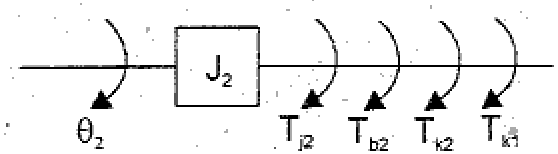


Fig 3.

By Newton's second law, $T_{j2} + T_{b2} + T_{k2} + T_{k1} = 0$

$$J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + K_2\theta_2) + K_1(\theta_2 - \theta_1) = 0 \qquad .....(2)$$

On replacing the angular displacements by angular velocity in the differential equations (1) and (2) governing the mechanical rotational system we get,

$$\left(\text{i.e., } \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}; \quad \frac{d\theta}{dt} = \omega \quad \text{and} \quad \theta = \int \omega\, dt\right)$$

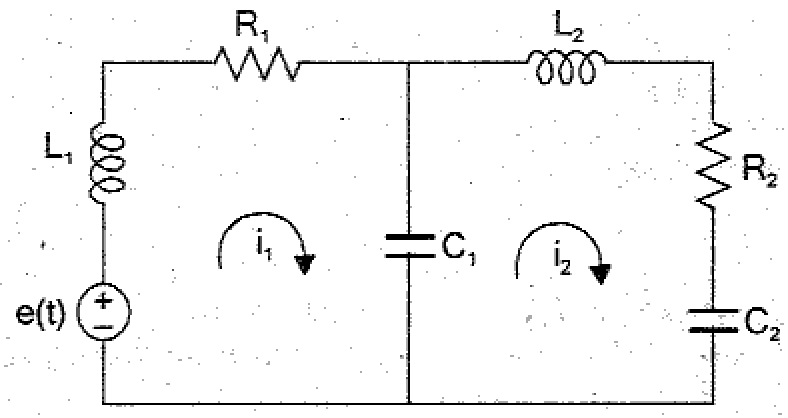
$$J_1 \frac{d\omega_1}{dt} + B_1\omega_1 + K_1\int(\omega_1 - \omega_2)dt = T \qquad .....(3)$$

$$J_2 \frac{d\omega_2}{dt} + B_2\omega_2 + K_2\int\omega_2 dt + K_1\int(\omega_2 - \omega_1)dt = 0 \qquad .....(4)$$

TORQUE-VOLTAGE ANALOGOUS CIRCUIT

The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow e(t)$	$J_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
$\omega_1 \rightarrow i_1$	$J_2 \rightarrow L_2$	$B_2 \rightarrow R_2$	$K_2 \rightarrow 1/C_2$
$\omega_2 \rightarrow i_2$			



The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 4 are given below (Refer fig 5 and 6).

$$L_1\frac{di_1}{dt}+R_1i_1+\frac{1}{C_1}\int(i_1-i_2)=e(t) \qquad .....(5)$$

$$L_2\frac{di_2}{dt}+R_2i_2+\frac{1}{C_2}\int i_2dt+\frac{1}{C_1}\int(i_2-i_1)dt=0 \qquad .....(6)$$

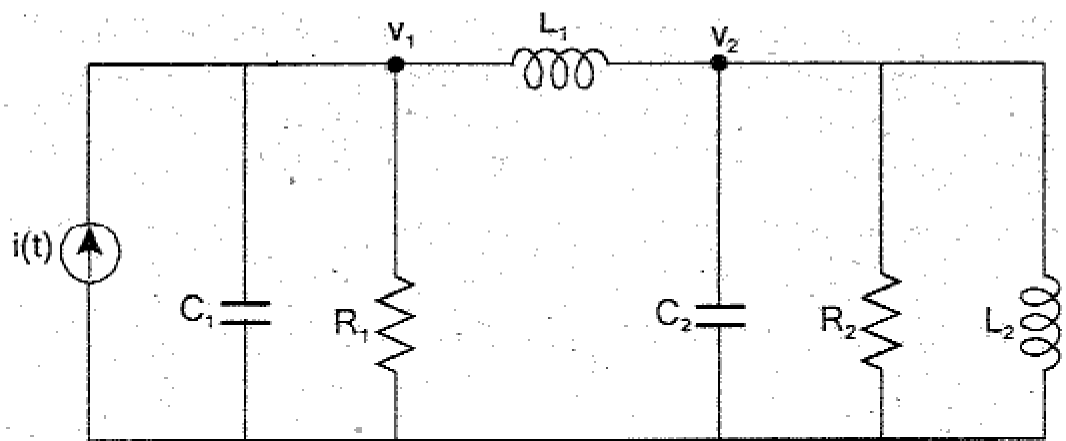
It is observed that the mesh basis equations (5) and (6) are similar to the differential equations (3) and (4) governing the mechanical system.

TORQUE-CURRENT ANALOGOUS CIRCUIT

The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow i(t)$ $\quad B_1 \rightarrow 1/R_1$ $\quad \omega_1 \rightarrow v_1$ $\quad J_1 \rightarrow C_1$ $\quad K_1 \rightarrow 1/L_1$

$B_2 \rightarrow 1/R_2$ $\quad \omega_2 \rightarrow v_2$ $\quad J_2 \rightarrow C_2$ $\quad K_2 \rightarrow 1/L_2$



The node basis equations using Kirchoff's current law for the circuit shown in fig 7 are given below (Refer fig 8 and 9).

$$C_1\frac{dv_1}{dt} + \frac{1}{R_1}v_1 + \frac{1}{L_1}\int(v_1 - v_2)dt = i(t) \qquad .....(7)$$

$$C_2\frac{dv_2}{dt} + \frac{1}{R_2}v_2 + \frac{1}{L_2}\int v_2 dt + \frac{1}{L_1}\int(v_2 - v_1)dt = 0 \qquad .....(8)$$

It is observed that the mesh basis equations (5) and (6) are similar to the differential equations (3) and (4) governing the mechanical system.

SERVO MOTORS :-

The motors that are used in the automatic control systems are called “servo motors”.

When the objective of the system is to control the position of an object, then the system is called as servo mechanism. Servo motors are used to convert an electric signal [controlled voltage] in to angular displacement of the shaft.

The servomotors should have the following requirements :-

1. Linear relationship between speed and electrical signal.
2. It should have linear speed torque characteristics.
3. The inertia of the rotor should be as low as possible.
4. The operation of the servo motor should be stable without any oscillation.
5. Fast response.
6. Wide range of speed control

Applications of servo motor

They are used in Radar system and process controller.

- Servomotors are used in computers and robotics.
- They are also used in machine tools.
- Tracking and guidance systems

CLASSIFICATION OF SERVO MOTORS :-

Based on the nature of electric supply, servo motors are classified as

- DC servo motors
- AC servo motors

DC SERVOMOTORS :-

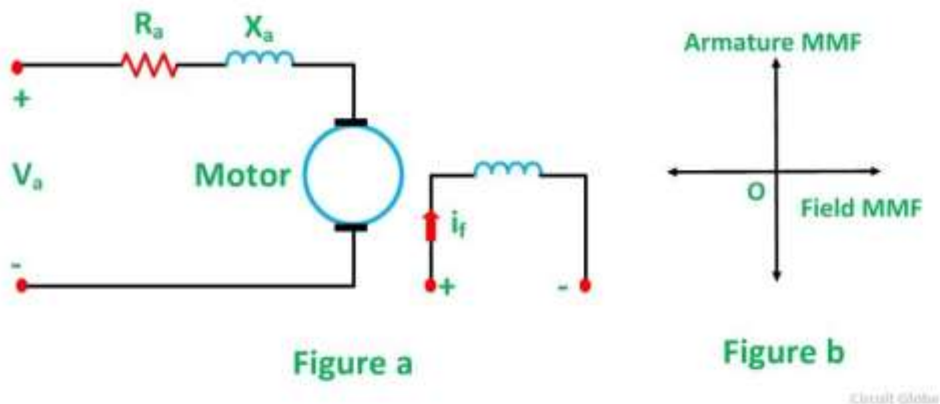
Basically dc servo motors are as same as normal motors. But there is one minor difference
between these two motors. i.e, all dc servo motors are essentially separately exited type

The advantages of dc servo motors are

1. Higher output than ac motor of same size.
2. Easier speed control from zero to full speed.
3. Higher torque to inertia ratio that gives quick response.
4. Dc servo motors have less weight.
5. The characteristics of dc servo motors are linear.
6. The dc servo motors are generally used for large power applications, such as machine
7. tools and robotics and so on.

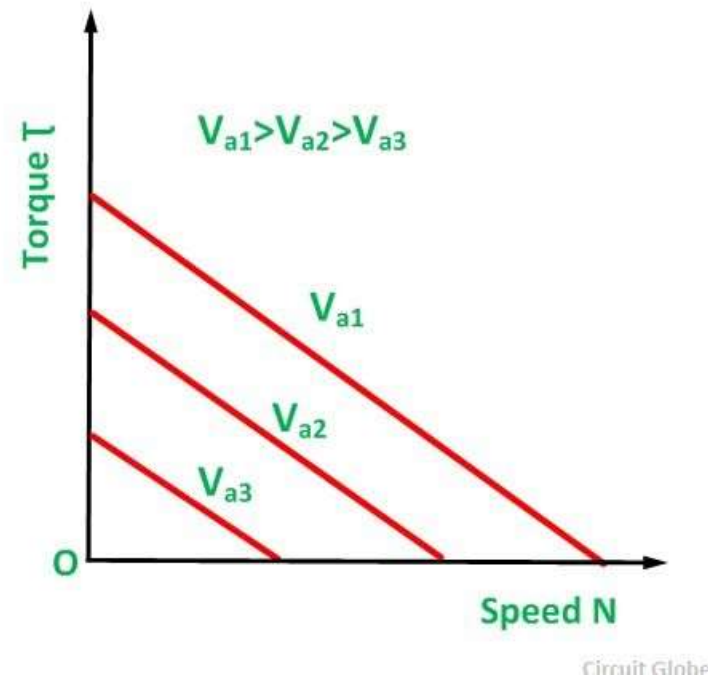
The figure (a) shows the connection of Separately Excited DC Servo motor

The figure (b) shows the armature MMF and the excitation d MMF in quadrature in a DC machine.



This provides a fast torque response because torque and flux are decoupled. Therefore, a small change in the armature voltage or current brings a significant shift in the position or speed of the rotor. Most of the high power servo motors are mainly DC.

The Torque-Speed Characteristics of the DC Servo Motor is shown below.



As from the above characteristics, it is seen that the slope is negative.

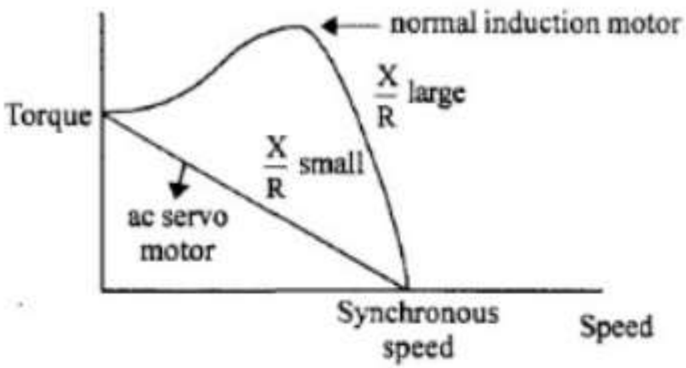
i. **Transfer function of field controlled dc servo motor.**
ii. **Transfer function of armature controlled dc servo motor.**

AC SERVOMOTORS:-

An Ac servo motor is basically a two phase induction motor except some design features.

A 2 Phase ac servo motor differs with normal induction motor in the following two ways.

1) The rotor of the servo motor is developed with high resistance, so that X/R ratio is low Which results in linear speed –torque characteristics. The induction motor has high X/R Ratio which results in high efficiency but non-linear speed –torque characteristics.



2) The excitation voltage applied to the 2 stator winding should have a phase difference of 90 degrees

CONSTRUCTION:-

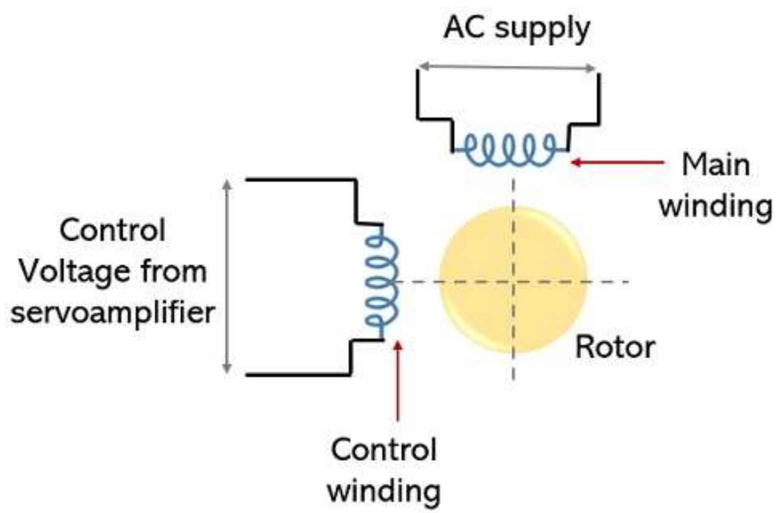
The main parts of ac servomotors are(1)Stator
(2)Rotor

(1)STATOR:

The stator contains two windings. They are(a)Main winding or reference winding
(b)control winding

A constant ac signal as input is provided to the main winding of the stator. The control winding is provided with the variable control voltage. This variable control voltage is obtained from the servo amplifier.

It is to be noted here that to have a rotating magnetic field, the voltage applied to the control winding must be 90° out of phase w.r.t the input ac voltage.



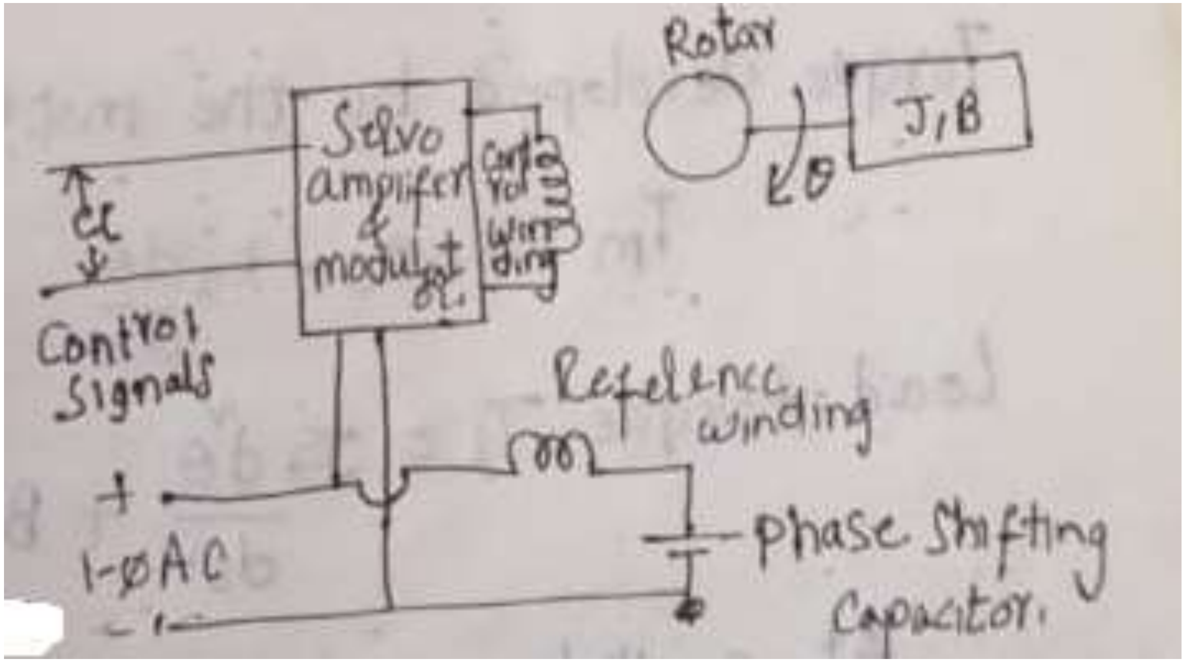
Stator of AC Servomotor

(2)ROTOR:-

The rotor usually Is of squirrel cage or drag-cup type. The squirrel cage rotor is made up of laminations. The rotor bars are placed in those slots and short circuited by end rings. The Diameter of this type of rotor is kept small in order to reduce inertia and to obtain good Accelerating characteristics.

The drag cup type rotor Is used in low power applications. The rotor will be in the form of hallow Cylinder made up of aluminium.The Al cylinder itself acts as short-circuited rotor conductor

SYMBOLIC REPRESENTATION OF AC SERVOMOTOR:



The control signals (e_c) in control systems are of low frequency, but in operation of servo motors, control phase voltage and reference phase voltage should have same frequency. Hence Control signal is modulated by a carrier whose frequency is equal to supply voltage.

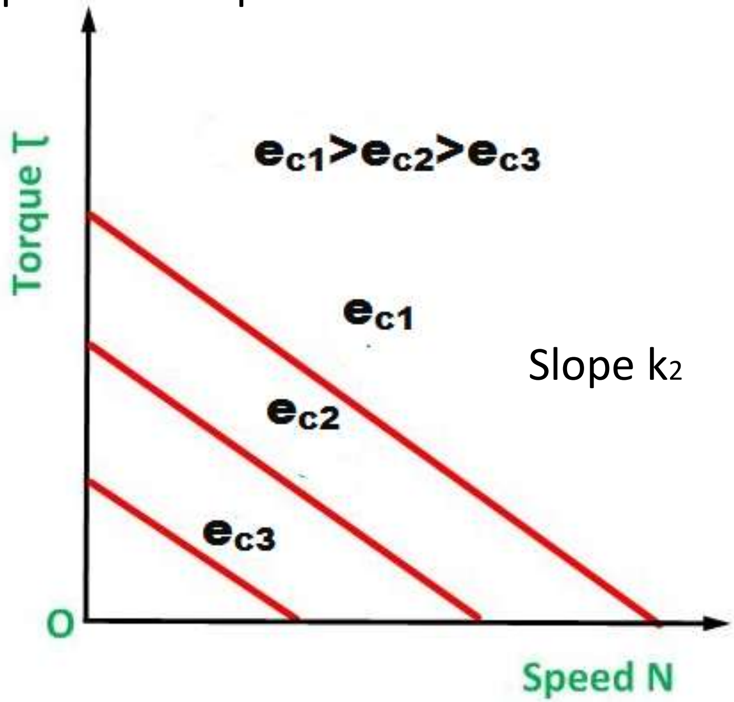
To maintain a phase shift of 90 degrees between control voltage and reference voltage a phase shift Capacitor is inserted in the reference winding.

Features

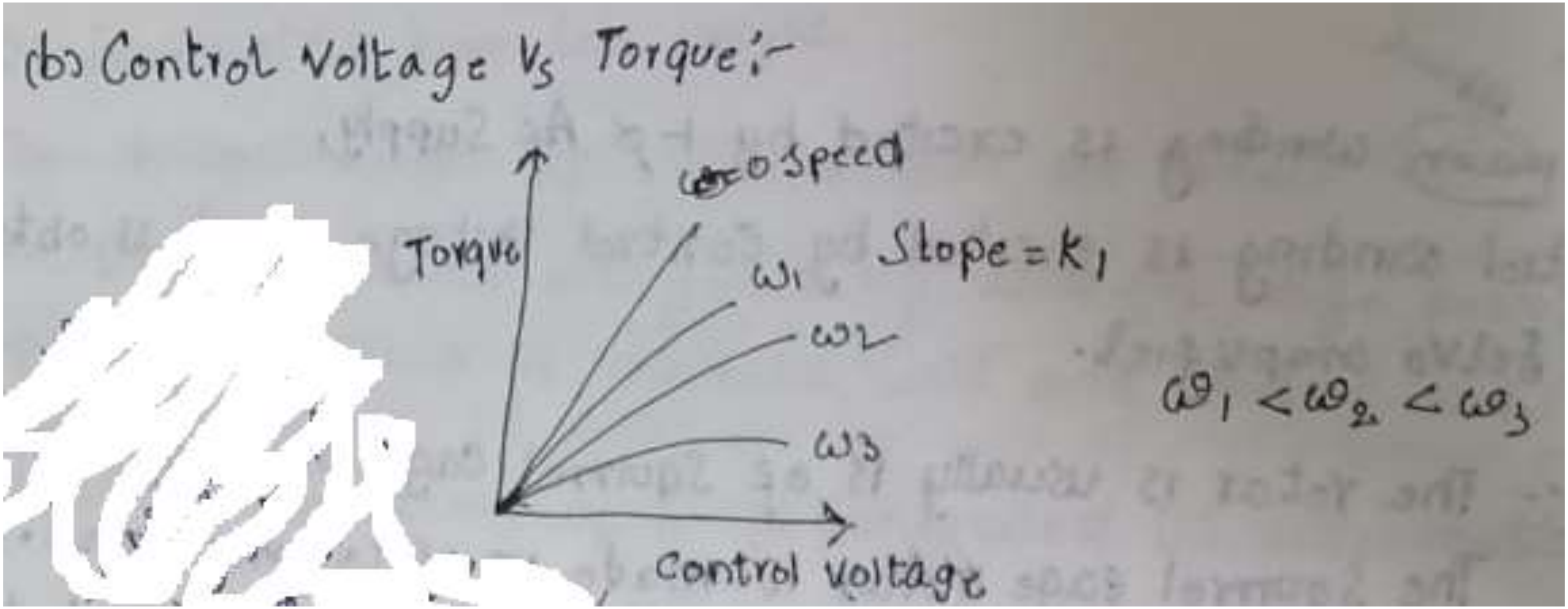
- ➢These are low weight devices.
- ➢It offers reliability as well as stability in operation.
- ➢There is not much noise generated at the time of operation.
- ➢It offers almost linear torque-speed characteristics.
- ➢As brushes and slip rings are not present here thus it reduces maintenance cost.

Ac servo motor characteristics:

1) Speed vs Torque



2. Control voltage vs torque



Transfer function of AC Servomotor.

T_m = Torque developed by AC Servomotor

T_L = Torque developed by Load.

J = Moment of inertia of load

B = dash point (frictional coefficient of rotar)

k_1 = Slope of control Voltage Vs torque characteristics

k_2 = Slope of Speed Vs torque characteristics.

Torque developed by the motar

$$T_m = k_1 e_c - k_2 \frac{d\theta}{dt}$$

Load torque $T_L = J * \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt}$

At equillibrium condition,

$$T_m = T_L$$

$$k_1 e_c - k_2 \frac{d\theta}{dt} = J * \frac{d^2\theta}{dt^2} + B * \frac{d\theta}{dt}$$

Taking Laplace transform on both sides

$$k_1 e_c(s) - k_2(s\,\theta(s)) = J(s^2\theta(s)) + Bs\,\theta(s)$$

$$k_1 e_c(s) = (sk_2 + Js^2 + Bs)\,\theta(s)$$

$$\frac{\theta(s)}{e_c(s)} = \frac{k_1}{Js^2 + (B + k_2)s} = \frac{\frac{K_1}{B+K_2}}{s\left(\frac{J}{B+K_2}s + 1\right)}$$

$$\frac{\theta(s)}{E(s)} = \frac{K_m}{s(1+s\tau_m)}$$

where $K_m = \frac{K_1}{B+K_2}$

= Motar Gain constant.

$\tau_m = \frac{J}{B+K_2}$

= Motar time constant.

Differen

Differences between A.C Servomotor and D.C Servo motor

A.C Servomotor	D.C. Servomotor
1) Low power output of about 0.5 W to 100 W.	1) Deliver high power output
2) Efficiency is less about 5 to 20 %.	2) High Efficiency.
3) Due to absence of commutator maintenance is less.	3) Frequent maintenance required due to commutator.
4) Stability problems are less.	4) More problems of stability.
5) No radio frequency noise.	5) Brushes produce radio frequency noise.
6) Relatively stable and smooth operation.	6) Noisy operation.
7) A.C. amplifier used have no drift.	7) Amplifiers used have a drift.